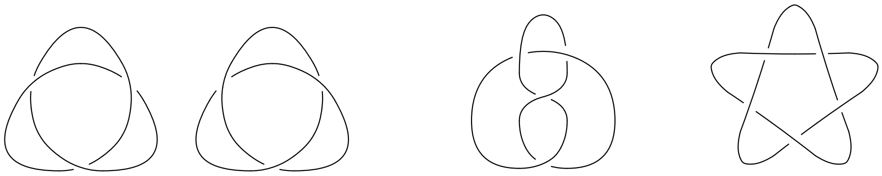
Polynomial Invariants for Knots

The determinant of a knot (HG/97–103) enables us to prove that knots exist—e.g., since the trefoil has a different determinant than the unknot, the trefoil is genuinely a knot. Remark 1 on HG/101 implies that there are infinitely many inequivalent knots; the sum of m trefoils will have determinant 3^m, which takes on different values for $m = 1, 2, \ldots$. The determinant also distinguishes the trefoil from the figure-eight.

On the other hand the determinant doesn't distinguish the trefoil from its mirror-image, as both have determinant 3. The figure-eight and the cinquefoil both have determinant 5, and there is a non-trivial knot which has determinant 1, the same as the unknot. We will develop some knot invariants more powerful than the determinant. They will be polynomials.



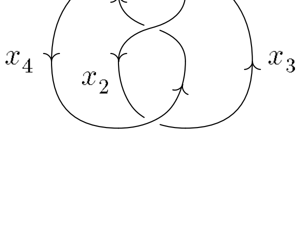
Trefoil Knot and mirror image Figure-8 knot Cinquefoil

The first is the Alexander polynomial, $A(t)$. The first step in computing the Alexander polynomial of a knot is a small modification of the first step in computing the determinant. We orient the knot, label its arcs $x_1, x_2, \ldots, x_n$ and construct an $n \times n$ matrix. Each row of the matrix, instead of containing $-1, -1, 2$ (and zeros), will contain $-1, t, 1-t$ (and zeros). First we have to distinguish between right-handed and left-handed crossings. Travelling on an overpass in the direction given by the orientation of the knot, if the underpass goes from right to left it's a right-handed crossing; if left to right it's a left-handed crossing.

right left

Now row i of the matrix will depend on what happens at the crossing where x_i is separated from x_{i+1}. Suppose these two arcs are separated by x_j. Then, if it's a right-hand crossing we let $a_{ii} = -1, a_{i,i+1} = t, a_{ij} = 1-t$; if it's a left-hand crossing we let $a_{ii} = t, a_{i,i+1} = -1, a_{ij} = 1-t$. Don't forget that for $i = n$ we interpret $i+1$ to mean 1.

For the figure-eight we find x_1 and x_2 are separated by x_3, at a right-hand crossing, so the top row of the matrix is $-1, t, 1-t, 0$.



Then x_2 and x_3 are separated by x_4 at a left-hand crossing, so the second row is $0, t, -1, 1-t$. Then x_3 and x_4 are separated by x_1, right-hand, and x_4 and x_1 are separated by x_2, left-hand, so the whole matrix is

$$\begin{pmatrix} -1 & t & 1-t & 0 \\ 0 & t & -1 & 1-t \\ 1-t & 0 & -1 & t \\ -1 & 1-t & 0 & t \end{pmatrix}.$$

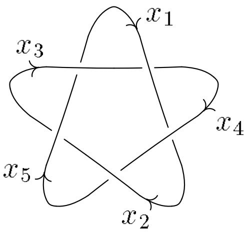
Now delete any row and any column from the matrix and compute the determinant of what's left; that's the Alexander polynomial of the knot. In our example let's delete the 4th column and first row; that leaves the matrix

$$\begin{pmatrix} 0 & t & -1 \\ 1-t & 0 & -1 \\ -1 & 1-t & 0 \end{pmatrix}$$

which has determinant $-1+3t-t^2$.

Now before we can be comfortable calling this the polynomial of the knot, we have to ask whether it depends on any of the choices we made along the way: the choice of which diagram to use for the knot, which orientation to take, which arc to call x_1 and which row and column to delete. It turns out that the polynomial does depend on the choices made but, if $A(t)$ and $B(t)$ are two polynomials obtained from a given knot by different choices, then $B(t) = \pm t^n A(t)$ for some integer n. The proof involves examining the effect of the Reidemeister moves on the matrix; we will not discuss it further. We note that we can always make the choices in such a way that $A(0) \neq 0$ and the leading coefficient is positive. With this normalization we have for the figure-eight knot $A(t) = t^2 - 3t + 1$.

Let's do the cinquefoil.



The crossings are 1|2–4R (meaning, x_1 is separated from x_2 by x_4 at a right-hand crossing), 2|3–5R, 3|4–1R, 4|5–2R, and 5|1–3R, leading to the matrix

$$\begin{pmatrix} -1 & t & 0 & 1-t & 0 \\ 0 & -1 & t & 0 & 1-t \\ 1-t & 0 & -1 & t & 0 \\ 0 & 1-t & 0 & -1 & t \\ t & 0 & 1-t & 0 & -1 \end{pmatrix}$$

Delete the last column and the third row (for example) and expand along the first column (for example) to obtain

$$\det\begin{pmatrix} -1 & t & 0 & 1-t \\ 0 & -1 & t & 0 \\ 0 & 1-t & 0 & -1 \\ t & 0 & 1-t & 0 \end{pmatrix} = -\det\begin{pmatrix} -1 & t & 0 \\ 1-t & 0 & -1 \\ 0 & 1-t & 0 \end{pmatrix} - t\det\begin{pmatrix} t & 0 & 1-t \\ -1 & t & 0 \\ 1-t & 0 & -1 \end{pmatrix}$$

which comes out to $t^4 - t^3 + t^2 - t + 1$. This proves that the figure-eight and cinquefoil are not equivalent; they have different Alexander polynomials.

Note that if, in setting up the matrix we let $t = -1$, we get exactly the matrix used in the computation of the determinant of a knot. It follows that if $A(t)$ is the Alexander polynomial of a knot K then $|A(-1)| = \det K$. Thus the Alexander polynomial is strictly an improvement on the determinant; any two knots that have different determinants must have different polynomials, but two knots with different polynomials may have the same determinant (as with the figure-eight and the cinquefoil).

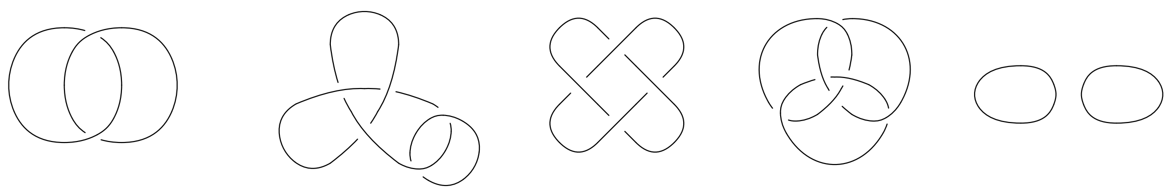
The next invariant we will discuss is the Conway polynomial. By multiplying by $\pm t^n$ for a suitable integer n, the Alexander polynomial can always be brought to a form which satisfies $A(t) = A(t^{-1})$ and $A(1) = 1$. For example the Alexander polynomial $t^2 - t + 1$ for the trefoil becomes $t - 1 + t^{-1}$; the Alexander polynomial $t^2 - 3t + 1$ for the figure-eight becomes $-t + 3 - t^{-1}$; the Alexander polynomial $t^4 - t^3 + t^2 - t + 1$ for the cinquefoil becomes $t^2 - t + 1 - t^{-1} + t^{-2}$ (the proof that this is always possible involves ideas not discussed in these notes). Let $z = \sqrt{t} - \frac{1}{\sqrt{t}}$ and notice that $z^2 = t - 2 + t^{-1}$, $z^4 = t^2 - 4t + 6 - 4t^{-1} + t^{-2}$. Then a bit of algebra shows that $A(t)$ can be expressed as a polynomial ∇ in z. This is the Conway polynomial. Thus the Conway polynomials for the trefoil, figure-eight and cinquefoil, respectively, are $z^2 + 1$, $-z^2 + 1$ and $z^4 + 3z^2 + 1$.

Since the Conway polynomial is just a change of variable in the (modified) Alexander polynomial, it is no better (and no worse) at distinguishing knots. But it has another property which makes it considerably easier to compute. Suppose that three oriented knots are identical except at one place, where the first knot (called $+$) has a right-hand crossing, the second (called $-$) has a left-hand crossing, and the third (called s for "smoothing") has no crossing. Then

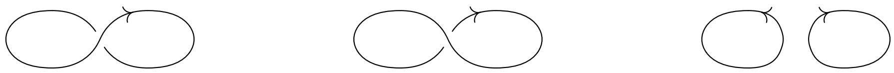
$$\nabla_+ \; - \; \nabla_- \;\; = \;\; -z\nabla_s \,.$$

The proof of this marvelous result involves ideas beyond the scope of these notes. The formula, together with $\nabla_{\text{unknot}} = 1$, gives us a simple way to compute the Conway polynomial of a knot, recursively. We will give many examples, but first we have to talk about links.

Just as a knot is a homeomorphic image in $\mathbf{R}^3$ of a circle, so a link is a homeomorphic image in $\mathbf{R}^3$ of a finite collection of circles. Here are some examples:

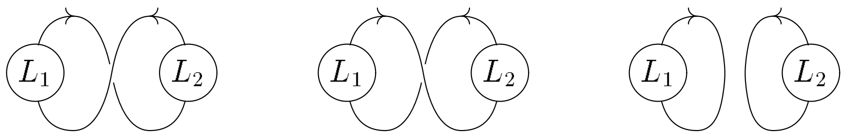


Let $+$ be the diagram on the left, below; it's a diagram of an unknot, with one (right-hand) crossing.



$+$ $\qquad$ $-$ $\qquad$ s

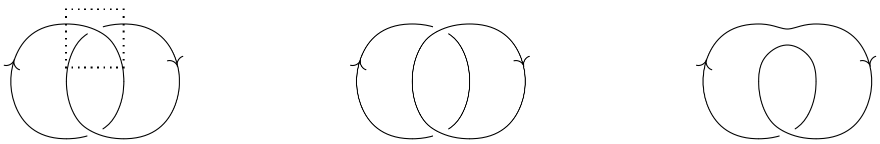
Then $-$ is also an unknot and s is an "unlink," two unlinked circles. By Conway's formula $\nabla_{\text{unknot}} - \nabla_{\text{unknot}} = -z\nabla_{\text{unlink}}$, whence $\nabla_{\text{unlink}} = 0$. In fact $\nabla_L = 0$ for any "split link"; that is, for any link which falls apart into 2 or more pieces. In the diagram below, L_1 and L_2 stand for arbitrary links.



$+$ $\qquad$ $-$ $\qquad$ s

It is clear that the $+$ and $-$ links are equivalent (just rotate L_1 through 360 degrees to go from one to the other), so $\nabla_+ = \nabla_-$, so $\nabla_s = 0$.

Let $+$ be the diagram on the left, below; it's a diagram of a simple right-hand link (let's call it, an SRL). The rectangle isn't part of the knot, it's just there to show you which crossing we're looking at.

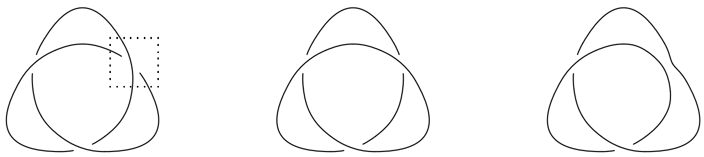


$+$ $\qquad$ $-$ $\qquad$ s

Then $-$ is an unlink and s is an unknot. So $\nabla_{\text{SRL}} - \nabla_{\text{unlink}} = -z\nabla_{\text{unknot}}$, whence $\nabla_{\text{SRL}} = -z$.

Note that orientation is important here. If we reverse the orientation on one of the circles in SRL to get a left-hand link (SLL), we find $\nabla_{\text{SLL}} = z$ (check it out!).

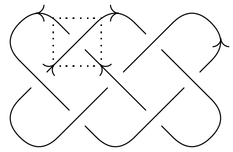
Now let's do the trefoil. Taking a trefoil with right-hand crossings as $+$, the $-$ diagram is the unknot and the s diagram is the SRL. So $\nabla_{\text{trefoil}} - \nabla_{\text{unknot}} = -z\nabla_{\text{SRL}}$ and $\nabla_{\text{trefoil}} = z^2 + 1$.



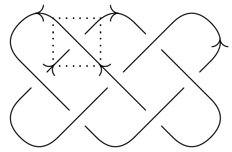
$+$ $\qquad$ $-$ $\qquad$ s

Let's do something more complicated, the knot labeled A below.

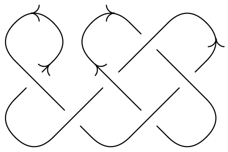
A



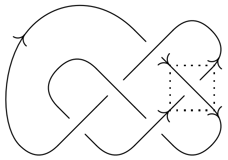
B



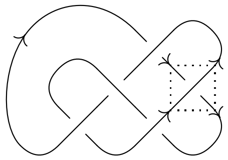
C



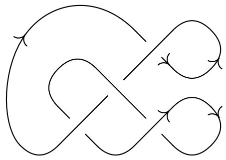
$D \sim B$



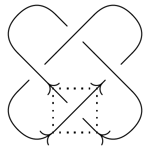
E



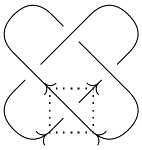
F



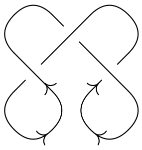
$G \sim C$



H



I



- A, B and C differ at only one crossing, where A is "right", B is "left" and C is "smooth"; so $\nabla_A - \nabla_B = -z\,\nabla_C$.
- D is obtained from B by a Type II Reidemeister move, so $\nabla_B = \nabla_D$.
- D, E and F differ at only one crossing, where D is "right", E is "left" and F is "smooth"; so $\nabla_D - \nabla_E = -z\,\nabla_F$. E is a trefoil so $\nabla_E = z^2 + 1$, and F is a simple right link so $\nabla_F = -z$.
- G is obtained from C by two Type I Reidemeister moves, so $\nabla_C = \nabla_G$.
- G, H and I differ at only one crossing, where G is "right", H is "left" and I is "smooth"; so $\nabla_G - \nabla_H = -z\,\nabla_I$. H is a simple right link so $\nabla_H = -z$, and I is the unknot so $\nabla_I = 1$.
- Putting it all together we get $\nabla_A = 4z^2 + 1$, which normalizes to $4t^2 - 7t + 4$ upon substituting $z = \sqrt{t} - \frac{1}{\sqrt{t}}$.

In January 1985, Vaughan Jones published a new polynomial invariant which was able to distinguish some knots with the same Alexander–Conway polynomial. Soon after, 5 different research groups found a two-variable polynomial which combines aspects of the Jones and Alexander–Conway polynomials and is stronger than either of them. It is called the *Homfly* polynomial — the name is taken from the initials of six of the discoverers, who published a joint paper three months after the publication of Jones' paper.

The *Homfly* polynomial of a knot (or link) K is denoted by $P_K(\ell, m)$ and is defined recursively by $P_{\text{unknot}}(\ell, m) = 1$, $\ell P_+ + \ell^{-1} P_- = -m P_s$. As before $+$, $-$, and s are knots that differ at just one crossing, where $+$ has a right crossing, $-$ has a left, and s has a "smoothing." That $P_K(\ell, m)$ is a knot invariant (that is, that it depends only on K and not on the choice of a diagram for K) is again beyond the scope of these notes. We will instead present some examples, following the calculations we did for the Conway polynomial two pages back.

Looking at the top set of diagrams on that page, we see that $\ell P_{\text{unknot}} + \ell^{-1} P_{\text{unknot}} = -m P_{\text{unlink}}$, but $P_{\text{unknot}} = 1$, so $P_{\text{unlink}} = -m^{-1}(\ell + \ell^{-1})$.

Going down to the third set of diagrams, we get $\ell P_{\text{SRL}} + \ell^{-1} P_{\text{unlink}} = -m P_{\text{unknot}}$. Putting in the known values for P_{unlink} and P_{unknot}, we find $P_{\text{SRL}} = -m\ell^{-1} + m^{-1}\ell^{-1} + m^{-1}\ell^{-3}$.

A similar calculation for SLL yields $P_{\text{SLL}} = -m\ell + m^{-1}\ell + m^{-1}\ell^{3}$.

Going down to the fourth set of diagrams and writing R for the right trefoil, we have $\ell P_R + \ell^{-1} P_{\text{unknot}} = -m P_{\text{SRL}}$, from which we deduce $P_R = -2\ell^{-2} - \ell^{-4} + m^2\ell^{-2}$.

If you work out the corresponding calculations for L, the left trefoil, you will find $P_L = -2\ell^2 - \ell^4 + m^2\ell^2$. Notice that $P_R \neq P_L$; the *Homfly* polynomial distinguishes between the left and right trefoils, which the Alexander–Conway polynomial cannot do.

The full story of the *Homfly* polynomial is not yet known. For example, no one has found a non-trivial knot K with $P_K = 1$ (examples of such things are known for the Alexander–Conway polynomial), nor has anyone proved that there are none.